2018-3-18

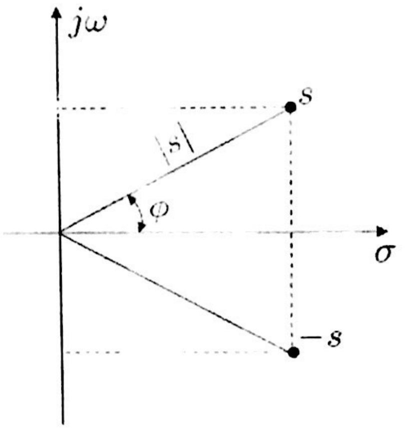
EE 362 Ch 2

Alternatively, we can represent the complex number s in polar form $s = |s|\, e^{j\phi}$.

where $|s| = \left(\sigma^2 + \omega^2\right)^{1/2}$ is the magnitude and $\phi \equiv \angle s = \operatorname{atan}(\omega/\sigma)$ the phase of s.

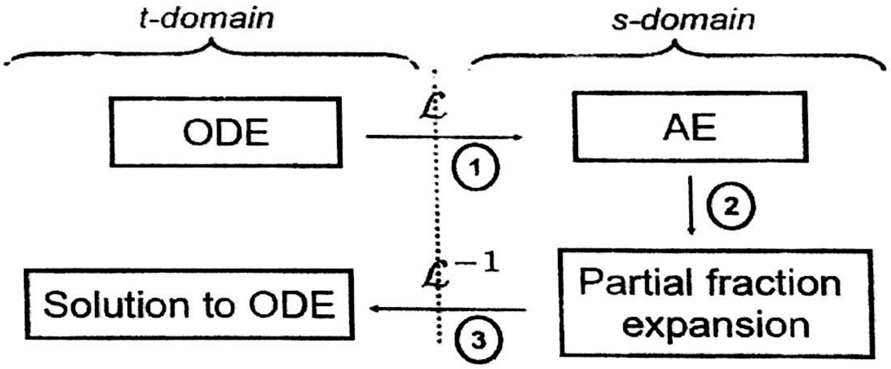
It is straightforward to derive

$$\frac{1}{s} = \frac{1}{|s|}e^{-j\phi} \Rightarrow \left|\frac{1}{s}\right| = \frac{1}{|s|} \qquad \text{and} \qquad \angle\frac{1}{s} = -\angle s.$$



An advantage of laplace transform:-

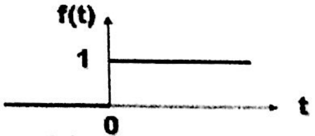
We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Examples of Laplace transform

- **Unit step function**

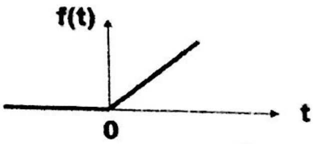
$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Longrightarrow F(s) = \int_0^\infty 1 \cdot e^{-st}dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \boxed{\frac{1}{s}}$$ *(Memorize this!)*

- **Unit ramp function**

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

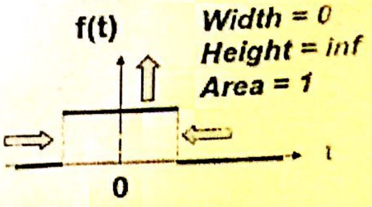


$$\Longrightarrow F(s) = \int_0^\infty t e^{-st}dt \underset{\uparrow}{=} -\frac{1}{s}\left[t e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \boxed{\frac{1}{s^2}}$$

(Integration by parts)

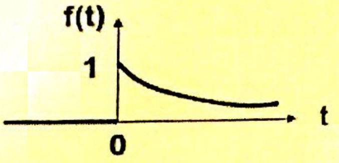
- Unit impulse function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \boxed{\int_{-\infty}^{\infty} \delta(t) g(t) dt = g(0)} \Rightarrow$$



$$\Longrightarrow \quad F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = e^{-s \cdot 0} = 1$$ *(Memorize this!)*

- Exponential function



$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Longrightarrow \quad F(s) = \int_0^{\infty} e^{-\alpha t} . e^{-st} dt = -\frac{1}{s+\alpha} \left[e^{-(s+\alpha)t} \right]_0^{\infty} = \frac{1}{s+\alpha}$$

- Sine function $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$

- Cosine function $\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$

(Memorize these!)

Remark:

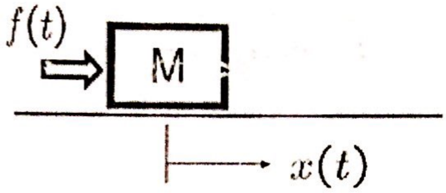
Instead of computing Laplace for each function, and/or transform for each function, memorizing complicated Laplace transform, use the Laplace transform table

!

Signal	Waveform $f(t)$	Transform F(s)
Impulse	$\delta(t)$	1
Step function	$u(t)$	$\frac{1}{s}$
Ramp	$tu(t)$	$\frac{1}{s^2}$
Exponential	$[e^{-\alpha t}]u(t)$	$\frac{1}{s+\alpha}$
Damped ramp	$[te^{-\alpha t}]u(t)$	$\frac{1}{(s+\alpha)^2}$
Sine	$[\sin \beta t]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$[\cos \beta t]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped sine	$[e^{-\alpha t} \sin \beta t]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$[e^{-\alpha t} \cos \beta t]u(t)$	$\frac{(s+\alpha)}{(s+\alpha)^2+\beta^2}$

Ex: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$



We want to know the trajectory of x(t). By Laplace transform,

$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

$$\Longrightarrow\ X(s) = \underbrace{\frac{1}{Ms^2}F(s)}_{} + \underbrace{\frac{x(0)}{s} + \frac{x'(0)}{s^2}}_{}$$

(Total response) = (Forced response) + (Initial condition response)

$$\Longrightarrow\ x(t) = \overbrace{\mathcal{L}^{-1}\left[\frac{1}{Ms^2}F(s)\right]}^{} + \overbrace{x(0)u_s(t) + x'(0)tu_s(t)}^{}$$

The Laplace transforms of some common functions of time are shown below

$$u_0(t) \Leftrightarrow 1/s$$

$$t \Leftrightarrow 1/s^2$$

$$t^n u_0(t) \Leftrightarrow \frac{n!}{s^{n+1}}$$

$$\delta(t) \Leftrightarrow 1$$

$$\delta(t-a) \Leftrightarrow e^{-as}$$

$$e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a}$$

$$te^{-at}u_0(t) \Leftrightarrow \frac{1}{(s+a)^2}$$

$$t^2e^{-at}u_0(t) \Leftrightarrow \frac{2!}{(s+a)^3}$$

$$t^ne^{-at}u_0(t) \Leftrightarrow \frac{n!}{(s+a)^{n+1}}$$

$$\sin\omega t\ u_0t \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

$$\cos\omega t\ u_0t \Leftrightarrow \frac{s}{s^2+\omega^2}$$

$$e^{-at}\sin\omega t\ u_0(t) \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}\cos\omega t\ u_0(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$$

Laplace transform theorem

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) \; = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] \quad = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] \quad = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] \quad = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] \quad = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] \quad = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] \quad = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] \quad = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) \quad = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) \quad = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

MATLAB

You can compute Laplace transform using the symbolic toolbox of MATLAB. If you want to compute the Laplace transform of *any function*, you can use the following MATLAB program.

f1(t)= 5 t^5 , f2(t)= exp(-a t) sin(3 t w) , f3(t)= 3 t exp(-5 t)

```
clc
syms w a t
f1=4*t^5;
disp('f1(t)=')
pretty(f1)
disp('F1(s)=')
F1=laplace(f1);
pretty(F1)
sprintf('\n')
```

```
          480
F1(s)=    ---
           6
          s
```

```
f2= exp(-a*t)*sin(3*w*t);
disp('f2(t)=')
pretty(f2)
disp('F2(s)=')
F2=laplace(f2);
pretty(F2)
sprintf('\n')
```

```
                  3 w
F2(s)=    ---------------
                 2       2
          (a + s)  + 9 w
```

```
f3=3*t*exp(-5*t);
disp('f3(t)=')
pretty(f3)
disp('F3(s)=')
F3=laplace(f3);
pretty(F3)
```

```
               3
F3(s)=    --------
                 2
          (s + 5)
```

EX: Calculating the Laplace F(s) transform of a function f(t) is quite simple in Matlab.

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

```
>> syms t s
>> f=-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t);
>> F=laplace(f,t,s)

F =

-5/4/s+7/2/(s+2)^2+5/4/(s+2)

>> simplify(F)

ans =

(s-5)/s/(s+2)^2

>> pretty(ans)
                                      s - 5
                                    ----------
                                             2
                                    s (s + 2)
```

which corresponds to F(s).

$$F(s) = \frac{(s-5)}{s(s+2)^2}$$

Alternatively, one can write the function f(t) directly as part of the laplace command:

```
>>F2=laplace(-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t))
```

2- Inverse Laplace Transform

The inverse Laplace transform can be obtained by use of the inversion integral given by the following equation

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds, \qquad \text{for } t > 0$$

The inversion integral is complicated and therefore, its use not recommended for finding inverse Laplace transforms To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called & ***partial-fraction expansion.***

Partial-Fraction Expansion

If $F(s) = \dfrac{N(s)}{D(s)}$ Where N(s) and D(s) are polynomials in s.

Note

I- If the order of N(s) is less than the order of D(s), then a partial-fraction expansion can be made.

II- If the order of N(s) is greater than or equal to the order of D(s), then N(s) must be divided by D(s) successively until the result has a remainder whose numerator is of order less than its denominator.

Case 1: Partial- fraction expansion when F(s) involves distinct poles only.

Ex: Find the inverse laplace transform of Y(s)

$$Y(s) = \frac{32}{s(s+4)(s+8)}$$

Poles:

p1= 0 p2= -4 p3 = -8

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

MATLAB

```
clc
syms s
num=32
den= poly([0 -4 -8])
[r,p,k]=residue(num,den)
g= r(1)/(s-p(1)) + r(2)/(s-p(2))+ r(3)/(s-p(3)) ;
pretty(g)
ilaplace(g)
```

$\Rightarrow \frac{num(s)}{den(s)}$ Partial Frac

```
 num =

      32


  den =

       1     12     32      0


   r =

       1
      -2
       1


   p =

      -8
      -4
       0


   k =

       []



    1         2      1
  ----- - ----- + -
  s + 8   s + 4    s

ans =

1/exp(8*t) - 2/exp(4*t) + 1
```

Case 2. Partial- fraction expansion when F(s) involves Real and Repeated poles

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)}$$

$$+ \frac{K_{r+1}}{(s+p_2)} + \cdots + \frac{K_n}{(s+p_n)}$$

$$F_1(s) = (s+p_1)^r F(s)$$

The general expression for K1 through Kr for the multiple roots is

$$K_i = \frac{1}{(i-1)!}\frac{d^{i-1}F_1(s)}{ds^{i-1}}\bigg|_{s\to -p_1} \qquad i = 1,2,\ldots,r; \quad 0! = 1$$

Ex: Find the inverse laplace transform of F(s)

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

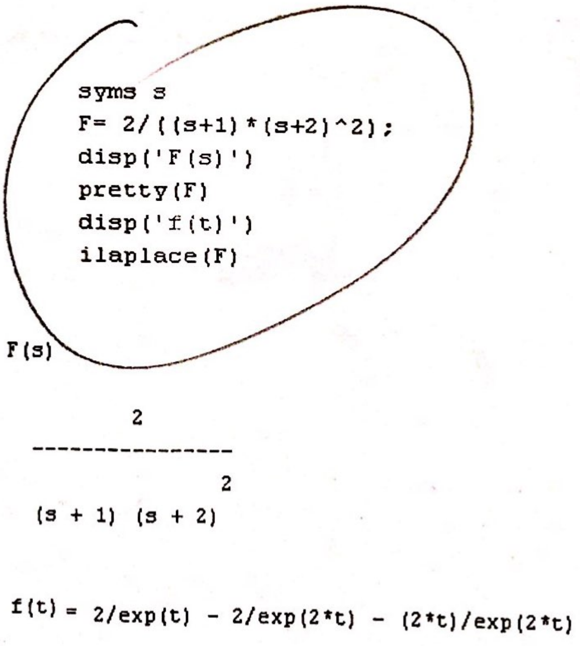
$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

$$K1 = (s+1).F(s) = \frac{2}{(s+2)^2} = 2$$

$$K2 = (s+2)^2.F(s) = \frac{2}{(s+2)^2} = -2$$

$$K3 = 1.\frac{d}{ds}\left((s+2)^2.F(s)\right) = \frac{2}{(s+1)} = 2$$

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$



```
syms s
F= 2/((s+1)*(s+2)^2);
disp('F(s)')
pretty(F)
disp('f(t)')
ilaplace(F)
```

```
F(s)

        2
-----------------
                2
(s + 1) (s + 2)

f(t) = 2/exp(t) - 2/exp(2*t) - (2*t)/exp(2*t)
```

Ex: Find the inverse laplace transform of F(s)

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + 1} + \frac{b_2}{(s + 1)^2} + \frac{b_3}{(s + 1)^3}$$

$$b_3 = \left[(s + 1)^3 \frac{B(s)}{A(s)}\right]_{s=-1}$$

$$= (s^2 + 2s + 3)_{s=-1}$$

$$= 2$$

$$b_2 = \left\{\frac{d}{ds}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1}$$

$$= \left[\frac{d}{ds}(s^2 + 2s + 3)\right]_{s=-1}$$

$$= (2s + 2)_{s=-1}$$

$$= 0$$

$$b_1 = \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1}$$

$$= \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2 + 2s + 3)\right]_{s=-1}$$

$$= \frac{1}{2}(2) = 1$$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s + 1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s + 1)^3}\right]$$

$$= e^{-t} + 0 + t^2 e^{-t}$$

$$= (1 + t^2)e^{-t}, \qquad \text{for } t \geq 0$$

Case 3: Partial- fraction expansion when F(s) involves complex or imaginary poles

Ex: Find the inverse laplace transform of F(s)

$$F(s) = \frac{1}{s(s^2+2s+2)}$$

$$s^2 + 2s + 2 = (s+1+j1)(s+1-j1)$$

$$F(s) = \frac{1}{s(s^2+2s+2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+2s+2}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{a_1(s^2+2s+2) + (a_2 s + a_3)s}{s(s^2+2s+2)}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{(a_1+a_2)s^2 + (2a_1+a_3)s + 2a_1}{s(s^2+2s+2)}$$

$$\therefore\ a_1 + a_2 = 0 \ \ \& \ \ 2a_1 + a_3 = 0 \ \ \& \ \ 2a_1 = 1$$

$$\therefore\ a_1 = \frac{1}{2}$$

$$a_2 = -a_1 = -\frac{1}{2}$$

$$a_3 = -2a_1 = -1$$

$$\frac{1}{s(s^2+2s+2)} = \frac{1/2}{s} - \frac{1}{2}\cdot\frac{s+2}{s^2+2s+2} = \frac{1/2}{s} - \frac{1}{2}\,\frac{s+2}{s^2+2s+2+1-1}$$

$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2+1^2}\right] - \frac{1}{2}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$f(t) = \left(\frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t\right)u(t)$$

Ex: Find the inverse laplace transform of F(s)

$$F_1(s)=\frac{s^3+2s^2+6s+7}{s^2+s+5} \qquad F_1(s)=s+1+\frac{2}{s^2+s+5}$$

$$F_2(s)=\frac{2}{S^2+S+5}=\frac{2}{S^2+S+5+\frac{1}{4}-\frac{1}{4}}$$

$$F_2(s)=\frac{2}{(S+\frac{1}{2})^2+\frac{19}{4}}=\frac{2}{(S+\frac{1}{2})^2+(\frac{\sqrt{19}}{2})^2}$$

$$\therefore F_1(s)=S+1+\frac{2}{\sqrt{19}}\,\frac{2*(\sqrt{19}/2)}{(S+\frac{1}{2})^2+(\sqrt{19}/2)^2}$$

$$=S+1+\frac{4}{\sqrt{19}}\cdot\frac{\sqrt{19}/2}{(S+\frac{1}{2})^2+(\frac{\sqrt{19}}{2})^2}$$

$$\begin{array}{r} S+1 \\ S^2+S+5\,\overline{)\,S^3+2S^2+6S+7} \\ \underline{S^3+S^2+5S} \\ S^2+S+7 \\ \underline{S^2+S+5} \\ 0+0+2 \end{array}$$

$$\therefore f(t)=\mathcal{L}^{-1}\left[F_1(s)\right]$$

$$=\frac{d}{dt}\delta(t)+\delta(t)+\frac{4}{\sqrt{19}}\cdot e^{-0.5t}\cdot\sin(\frac{\sqrt{19}}{2}t)$$

$$=\frac{d}{dt}\delta(t)+\delta(t)+\frac{4\sqrt{19}}{19}\,e^{-0.5t}\,\sin(\frac{\sqrt{19}}{2}t)$$

```
clc
syms s
gs= s + 1 +( 2/(s^2+s+5));
gt= ilaplace(gs);
gt=simple(gt);
c
!
                                                       /    1/2      \
                                  1/2      /   t \     |  19     t   |
                              4 19      exp| - - | sin| ------- |
                                           \   2 /     \    2    /
     g(t)=   dirac(t) + dirac(t, 1) + ------------------------------------
                                                      19
```

* MATLAB Function [heaviside, dirac]

EX: Find the inverse laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 5}{s(s+1)}$$

$$F(s) = s^2 + s + 2 + \frac{2s+5}{s(s+1)}$$

$$= s^2 + s + 2 + \frac{a_1}{s} + \frac{a_2}{s+1}$$

$$a_1 = \left.\frac{2s+5}{s+1}\right|_{s=0} = 5$$

$$a_2 = \left.\frac{2s+5}{s}\right|_{s=-1} = -3$$

$$\begin{array}{r|l} & s^2 + s + 2 \\ \hline s^2 + s & s^4 + 2s^3 + 3s^2 + 4s + 5 \\ & s^4 + s^3 \\ \hline & s^3 + 3s^2 + 4s + 5 \\ & s^3 + s^2 \\ \hline & 2s^2 + 4s + 5 \\ & 2s^2 + 2s \\ \hline & 2s + 5 \end{array}$$

$$F(s) = s^2 + s + 2 + \frac{5}{s} + \frac{3}{s+1}$$

$$f(t) = \frac{d^2}{dt}\delta(t) + \frac{d}{dt}\delta(t) + 2\delta(t) + 5 - 3e^{-t}, \text{ for } t \geq 0$$

MATLAB

```
num = [1 2 3 4 5]
den = [ 1 1 0]
[r, p, k] = residue(num, den)
r = -3        p = -1        k = 1    1    2
     5             0
```

Laplace Transform Solution of a Differential Equation

EX: Given the following differential equation, solve for *y(t)* if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t)$$

$$S^2Y(s) + SY(0) - Y(0) + 12\,(SY(s) - Y(0)) + 32Y(s) = 32/s$$

if all initial conditions are zero

$$(S^2 + 12S + 32)Y(s) = 32/s$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

MATLAB

```
syms s
Y= 32/(s*(s^2 +12*s+32));
disp('Y(s)=')
pretty(Y)

yt=ilaplace(Y);
disp('y(t)')
pretty(Gt)
```

```
                    32
Y(s)=    ------------------
                2
          s (s  + 12 s + 32)

y(t) =  1 + exp(-8 t) - 2 exp(-4 t)
```

Example Solve using the Laplace transform

$$y' - y = e^{3t}, \quad y(0) = 2.$$

Application of the Laplace transform leads to

$$sY(s) - y(0) - Y(s) = \frac{1}{s-3},$$

therefore,

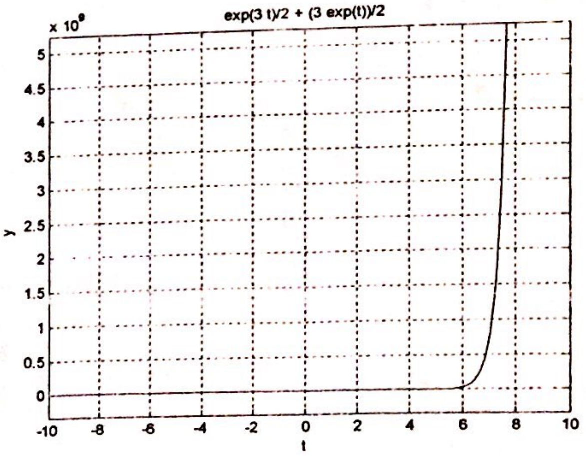
$$Y(s) = \frac{2}{s-1} + \frac{1}{(s-1)(s-3)} = \frac{2}{s-1} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s-3}\right).$$

Using the table to find the inverse Laplace transform, we obtain

$$y(t) = \mathscr{L}^{-1}\{Y\} = 2e^{t} - \frac{1}{2}(e^{t} - e^{3t}) = \frac{3}{2}e^{t} + \frac{1}{2}e^{3t}.$$

MATLAB

```
clc
syms y t
yt=dsolve('Dy-y=exp(3*t)','y(0)=2');
disp('y(t)=')
pretty(yt)
ezplot(yt,[-10 10])
grid
xlabel('t')
ylabel('y')
```



Example . Solve

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1,\ y'(0) = 0.$$

Applying the Laplace transform, I find

$$(s^2 - 3s + 2)Y - s + 3 = \frac{1}{s-3} \implies Y(s) = \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)}.$$

$$Y(s) = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1}.$$

To find A, B and C here is especially simple. For example, for A multiply both sides by $s-3$ and plug $s = 3$ into the expressions to obtain $A = \frac{1}{2}$. In a similar way $B = -2$ and $C = \frac{5}{2}$.

Therefore, using the linearity of the inverse Laplace transform, we will find

$$y(t) = \mathscr{L}^{-1}\{Y\} = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}.$$

Example . Solve

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1,\ y'(0) = 2.$$

Applying the Laplace transform to both side, we find

$$(s^2 - 10s + 9)Y + s - 2 - 10 = \frac{5}{s^2} \implies Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}.$$

To find the inverse Laplace transform we will need first simplify the expression for $Y(s)$ using the partial fraction decomposition:

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}.$$

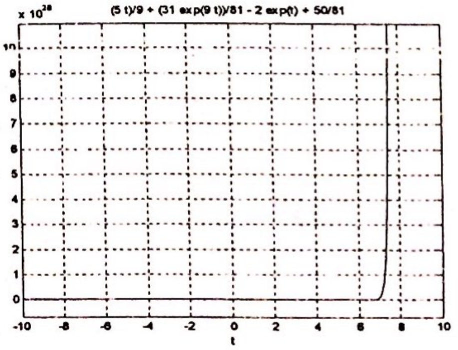
We find

$$B = \frac{5}{9}, D = -2, C = \frac{31}{81}, A = \frac{50}{81}.$$

Therefore, using the linearity of the inverse Laplace transform,

$$y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^{t}.$$

```
clc
syms y t
yt=dsolve('D2y-10*Dy+9*y=5*t','Dy(0)=2','y(0)=-1');
disp('y(t)=')
pretty(yt)
ezplot(yt,[-10 10])
grid
xlabel('t')
ylabel('y')
```



Example . Solve

$$y'' - 6y' + 15y = 2\sin 3t, \quad y(0) = -1,\ y'(0) = -4.$$

We have

$$(s^2 - 6s + 15)Y + s - 2 = \frac{6}{s^2+9} \Longrightarrow Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}.$$

To find the constants, we need to simplify the expression on the right (find the common denominator) and equate the coefficients at the equal powers:

$$\begin{aligned} s^3&: A + C = -1 \\ s^2&: -6A + B + D = 2 \\ s^1&: 15A - 6B + 9C = -9 \\ s^0&: 15B + 9D = 24 \end{aligned}$$

The solution is

$$A = \frac{1}{10}, B = \frac{1}{10}, C = -\frac{11}{10}, D = \frac{5}{2}.$$

Hence, we got

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right).$$

Now we need to find the inverse Laplace transform. Let us start with the first term:

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \cos 3t + \frac{1}{3}\sin 3t.$$

The second term is slightly more involved. Rearrange the expression in the following way (remember that we can always add and subtract the same expression and multiply and divide by the same expression different from zero):

$$\begin{aligned} \frac{-11s+25}{s^2-6s+15} &= \frac{-11s+25}{(s-3)^2+6} \\ &= \frac{-11(s-3)-8}{(s-3)^2+6} \\ &= -11\frac{(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}. \end{aligned}$$

Now

$$\mathscr{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = -11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t.$$

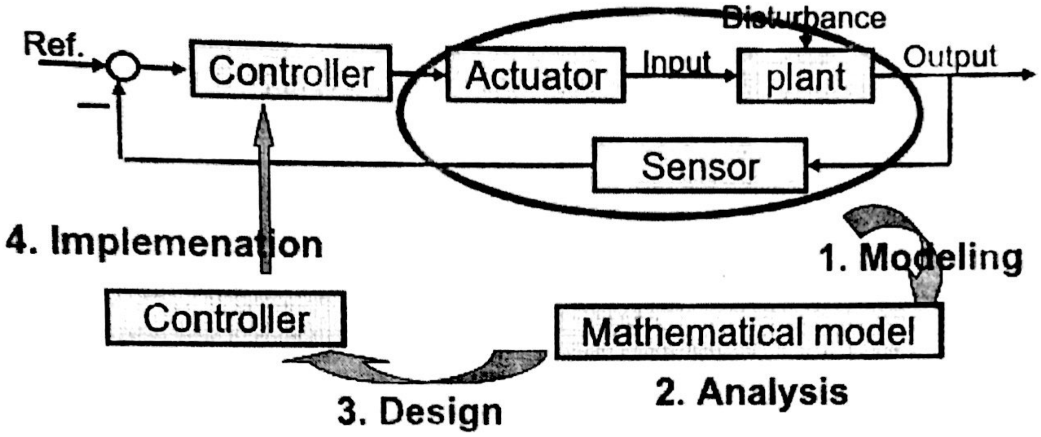
The final answer hence is

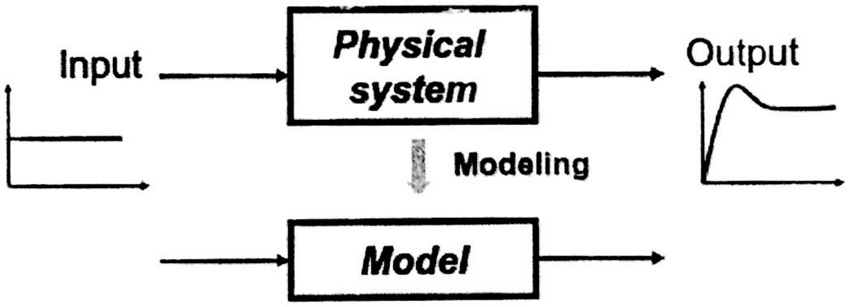
$$y(t) = \mathscr{L}^{-1}\{Y\} = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t\right).$$

Week #2 (cont)

Modelling in the Frequency Domain EE362

Mathematical model: Representation of the input- output (signal) relation of a physical system





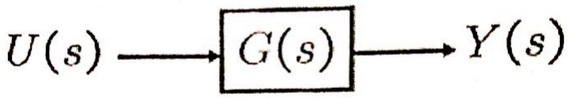
Remarks

- A model is used for the analysis and design of control systems.
- Modeling is the most important and difficult task in control system design.
- No mathematical model exactly represents a physical system.
- Do not confuse models with physical systems.
- In this course, we may use the term system to mean a mathematical model.

Math model ≠ Physical system
Math model ≈ Physical system

Transfer function: A transfer function is defined by

$$G(s) := \frac{Y(s)}{U(s)}$$



The transfer function of a **Linear**, **Time Invariant** LTI, differential equation system is defined as the ration of the Laplace transform of the output to the Laplace transform of the input **under the assumption that all initial condition are zero.**

Let us begin by writing a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

Where c(t) is the output, r(t) is the input, and the a_i's, b_i's, and the form of the differential equation represent the system. Taking the Laplace transform of both sides,

$$\begin{aligned} & a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t) \\ &= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t) \end{aligned}$$

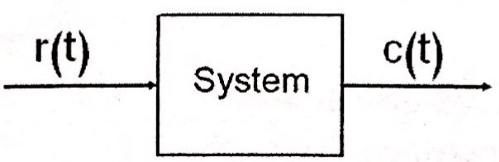
If we assume that all initial conditions are zero,

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$

Now form the ratio of the output transform, C(s), divided by the input transform, R(s):

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}}$$

We call this ratio, G(s), the **Transfer Function** and evaluate it with **zero initial conditions.**



C(s) =G(s).R(s)

Definition:

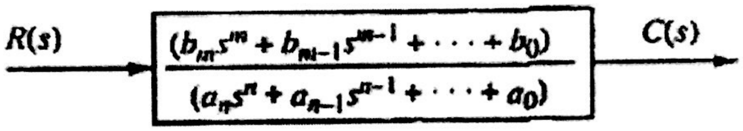
Poles of G(s) = roots of the denominator of G(s)

= roots of **characteristic equation**

Zeros of G(s)= roots of the numerator of G(s)

Stable: A system G(s)is said to be stable if the output c(t) of the system with a bounded input r(t) is bounded.

The transfer function can be represented as **a block diagram**, as shown in Figure below, with the input on the left, the output on the right, the system transfer function inside the block.


$R(s)$
$\frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$
$C(s)$

A **block diagram** of a system is a pictorial representation of the functions performed by each component and of the follow signals.

Summary

To derive the transfer function, we proceed according to the following steps:

1- Write the differential equation for the system.

2- Take the Laplace transform of the differential equation, assuming all initial condition are zero.

3- Take the ratio of the output C(s) to the input R(s). This ratio is the transfer function.

We can find the output, C(s) by using

$$C(s) = G(s) \, . \, R(s)$$

C(t) = inverse lablace (C(s)) = Response of the system to the input r(t).

If input = step function output (*step response*) = c(t) ,

If input = delta function output (*impulse response*) = c(t) ,

If input = ramp function output (*ramp response*) = c(t)

Some standard forms for transfer function

1- Ratio of polynomials

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

2- Factored forms or product form

$$H(s) = K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \begin{matrix} \leftarrow \text{zeros} \\ \\ \leftarrow \text{poles} \end{matrix}$$

3- Come form : comes from the partial fraction expansion

$$H(s) = K + \sum_{i=1}^{n} \frac{\alpha_i}{s - p_i}$$

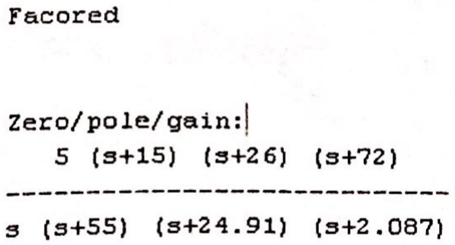
MATLAB: Use MATLAB to generate the transfer function:

$$G(s) = \frac{5(s+15)(s+26)(s+72)}{s(s+55)(s^2+27s+52)}$$

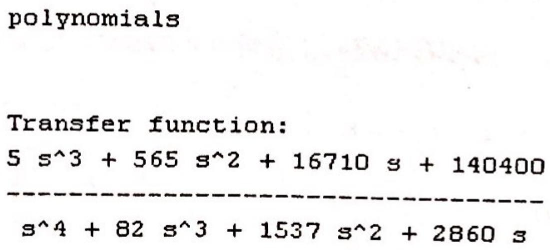
In the following ways:
a- The ratio of factors

b- The ratio of polynomials

```
'Factored'
Gp= zpk([ -15 -26 -72],[0 -55 roots([ 1 27 52])'],5)
'polynomials'
G= tf(Gp)
```



```
Facored


Zero/pole/gain:
   5 (s+15) (s+26) (s+72)
----------------------------
s (s+55) (s+24.91) (s+2.087)
```



```
polynomials


Transfer function:
5 s^3 + 565 s^2 + 16710 s + 140400
----------------------------------
 s^4 + 82 s^3 + 1537 s^2 + 2860 s
```

Ex Use MATLAB to generate the transfer function in the following form (factors form)

$$G(s) = \frac{s^3 + 9s^2 + 23s + 15}{s^4 + 20s^3 + 140s^2 + 400s + 384}$$

```
clc
num= [1 9 23 15];
den=[ 1 20 140 400 384];
disp(' ratio form Gr(s) =')
Gr= tf(num,den) % ratio

[z,p,k]= tf2zp(num,den)
disp(' factors form Gf(s) =')
Gf= zpk(z,p,k)  % facot
```

```
ratio form Gr(s)  =

Transfer function:
      s^3 + 9 s^2 + 23 s + 15
-------------------------------------
s^4 + 20 s^3 + 140 s^2 + 400 s + 384


z =                 p =                  k =

   -5.0000             -8.0000                 1
   -3.0000             -6.0000
   -1.0000             -4.0000
                       -2.0000


    factors form Gf(s)  =

   Zero/pole/gain:
      (s+5)  (s+3)  (s+1)
   ------------------------
   (s+8)  (s+6)  (s+4)  (s+2)
```

Ex Use MATLAB to generate the transfer function in the following form (ratio form)

$$G(s) = \frac{(s+1)(s+3)(s+5)}{(s+2)(s+4)(s+6)(s+8)}$$

```
num= poly([-1 -3 -5]);
den= poly([-2 -4 -6 -8])
disp(' ratio form Gr(s) =')
Gr= tf(num,den) % ratio
```

```
Transfer function:
      s^3 + 9 s^2 + 23 s + 15
-------------------------------------
s^4 + 20 s^3 + 140 s^2 + 400 s + 384
```

MATLAB

Given the zero(s), pole(s), and gain K of $B(s)/A(s)$, obtain the function $B(s)/A(s)$ using MATLAB. Consider the three cases below.

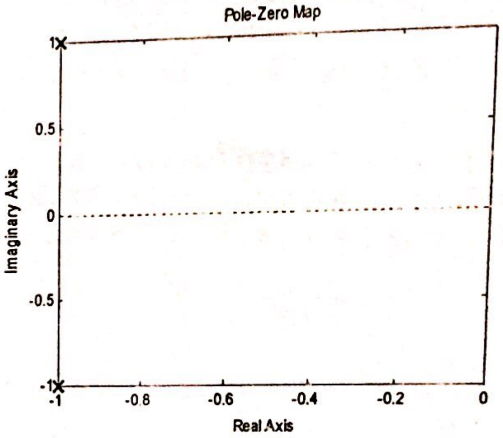
(1) There is no zero. Poles are at $-1+2j$ and $-1-2j$. $K = 10$.

(2) A zero is at 0. Poles are at $-1+2j$ and $-1-2j$. $K = 10$.

(3) A zero is at -1. Poles are at $-2, -4$ and -8. $K = 12$.

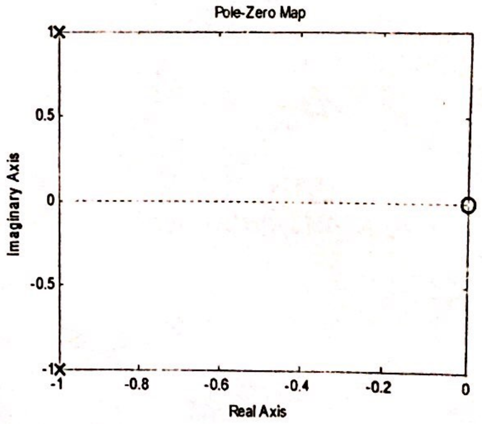
```
z=[];
p=[-1+2j ; -1-2j];
k=10;
[num,den]=
zp2tf(z,p,k);
G= tf(num,den)
pzmap(num,den)
```

```
Transfer function:
      10
--------------
s^2 + 2 s + 5
```



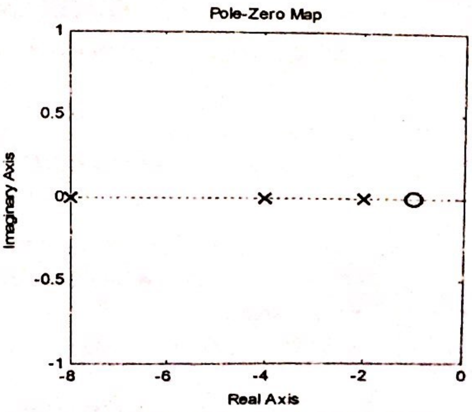
```
z=[0];
p=[-1+2j ; -1-2j];
k=10;
[num,den]=
zp2tf(z,p,k);
G= tf(num,den)
pzmap(num,den)
```

```
Transfer function:
      10 s
--------------
s^2 + 2 s + 5
```



```
z=[-1];
p=[-2 ; -4 ; -8];
k=12;
[num,den]=
zp2tf(z,p,k);
G=tf(num,den)
pzmap(num,den)
```

```
Transfer function:
          12 s + 12
---------------------------
s^3 + 14 s^2 + 56 s + 64
```



Ex

Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Taking the Laplace transform of both sides, ***assuming zero initial conditions***, we have

$$sC(s) + 2C(s) = R(s)$$

The transfer function, G(s), is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Ex

Use the result in EX1 to find the response, c(t) to inputs, assuming zero initial conditions.

a- r(t) = u(t), unit step b- r(t) = delta(t), impulse c- r(t) = t, ramp

a- r(t) = step function R(s) = 1/s

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

$$C(s) = \frac{1}{s+2}.\frac{1}{s} = \frac{1}{s(s+2)}$$

Expanding by partial fractions, we get

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

b- r(t) = impulse function R(s) = 1

$$C(s) = G(s).R(s)$$

$$C(s) = \frac{1}{s+2}.R(s) = \frac{1}{s+2}$$

$$c(t) = \exp(-2t)$$

c- r(t) = t ,ramp function R(s) = $1/s^2$

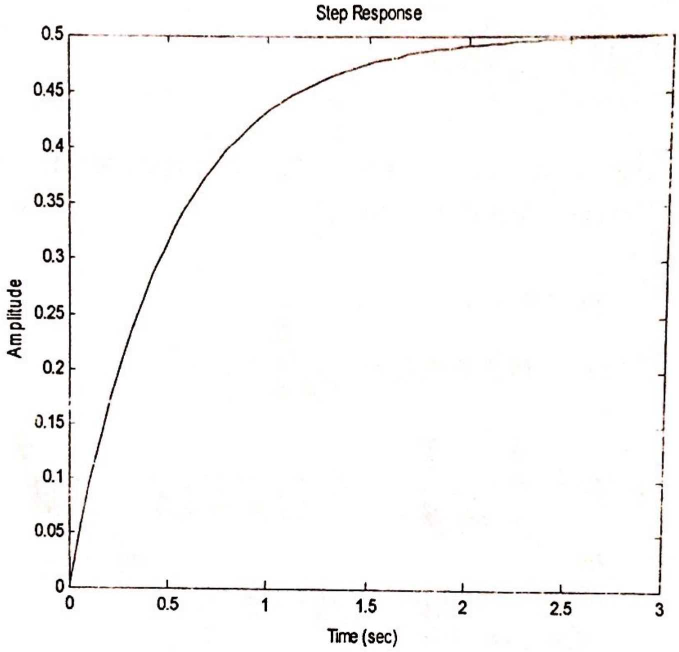
$$C(s) = \frac{1}{s+2}.\frac{1}{s^2} = \frac{1}{s^2(s+2)}$$

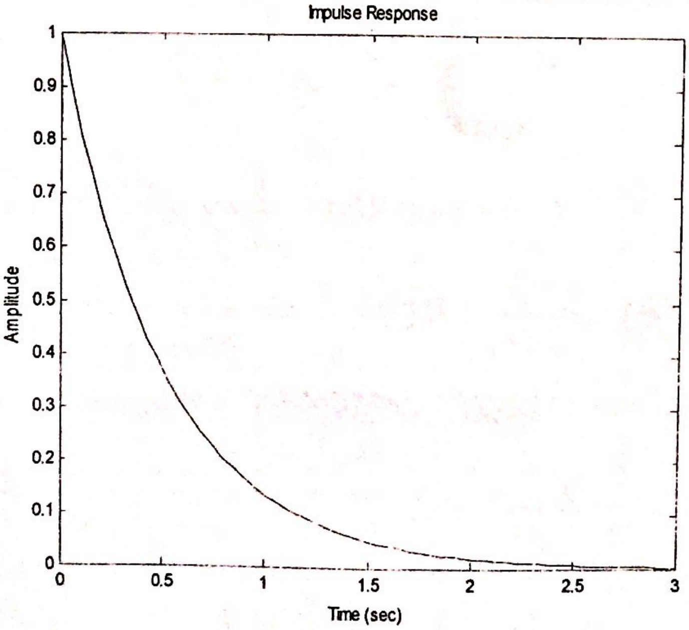
Expanding by partial fractions, we get

$$C(s) = \frac{1/4}{s+2} + \frac{1/2}{s^2} - \frac{1/4}{s}$$

$$c(t) = -0.25 \quad + 0.5t + 0.25\exp(-2t)$$

```
G= tf(1,[1 2])
step(G)
figure
impulse(G)
```





Ex

Find the transfer function, G(s) = C(s)/R(s), corresponding to the differential equation

$$\frac{d^3c}{dt^3}+3\frac{d^2c}{dt^2}+7\frac{dc}{dt}+5c=\frac{d^2r}{dt^2}+4\frac{dr}{dt}+3r.$$

Taking the Laplace transform of the differential equation assuming zero initial conditions yields:

$$s^3C(s)+3s^2C(s)+7sC(s)+5C(s)=s^2R(s)+4sR(s)+3R(s)$$

Collecting terms,

$$(s^3+3s^2+7s+5)C(s)=(s^2+4s+3)R(s)$$

Thus,

$$\frac{C(s)}{R(s)}=\frac{s^2+4s+3}{s^3+3s^2+7s+5}$$

Ex

Use the result in EX3 to find the response, c(t) to an input, r(t) = delta(t), a impulse function, assuming zero initial conditions.

To solve the problem, we use the equation

$$C(s) = G(s).R(s)$$

Where

$$\frac{C(s)}{R(s)}=\frac{s^2+4s+3}{s^3+3s^2+7s+5} \quad = \quad G(s)$$

C(s) = R(s) . G(s)

Since r(t) = u(t) step function , R(s) = 1,

$$C(s)=\frac{s^2+4s+3}{s^3+3s^2+7s+5}.R(s)$$

$$=\frac{s^2+4s+3}{(s^3+3s^2+7s+5)}C(s)=\frac{s^2+4s+3}{s^3+3s^2+7s+5}.1$$

c(t) = inverse laplace of C(s)

Note: Using a hand calculation, find the inverse Laplace transform of $C(s)$.

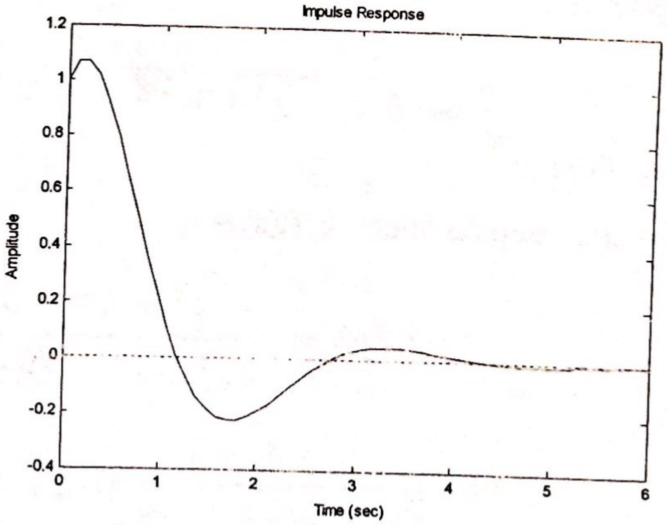
Program

```
clc
syms s
G= (s^2+4*s+3)/(s^3 +3*s^2 +7*s+5);
disp('G(s) = ')
pretty(G)

R= 1  ;  % where r(t) = delta function
C= R * G;
disp('C(s)= ')
pretty(C)

disp('c(t)= ')
c= ilaplace(C);
pretty(c)
```

```
G(s) =

          2
         s  + 4 s + 3
    --------------------
     3      2
    s  + 3 s  + 7 s + 5

C(s)=

          2
         s  + 4 s + 3
    --------------------
     3      2
    s  + 3 s  + 7 s + 5


c(t)=

  exp(-t) (cos(2 t) + sin(2 t))
```

Program

```
G= tf([1 4 3],[1 3 7 5])
impulse(G)
```



Ex5

Repeat the EX3 if the input r(t) = u(t), step function.

$$C(s) = G(s).R(s)$$

Where

$$\frac{C(s)}{R(s)} = \frac{s^2+4s+3}{s^3+3s^2+7s+5} \quad = G(s)$$

C(s) = R(s) . G(s)

Since r(t) = u(t) step function , R(s) = 1/s,

$$C(s) = \frac{s^2+4s+3}{s^3+3s^2+7s+5}.R(s)$$

$$= \frac{s^2+4s+3}{s(s^3+3s^2+7s+5)} C(s) = \frac{s^2+4s+3}{s^3+3s^2+7s+5}\cdot\frac{1}{s}$$

c(t) = inverse laplace of C(s)

Note: Using a hand calculation, find the inverse Laplace transform of C(s).

Program

```
clc
syms s
G= (s^2+4*s+3)/(s^3 +3*s^2 +7*s+5);
disp('G(s) = ')
pretty(G)

R= 1/s   ;   % where r(t)= step function
C= R * G;
disp('C(s)= ')
pretty(C)

disp('c(t)= ')
c= ilaplace(C);

pretty(c)
```

```
G(s) =

       2
      s  + 4 s + 3
  --------------------
   3      2
  s  + 3 s  + 7 s + 5

C(s)=

          2
         s  + 4 s + 3
  -----------------------
       3      2
  s (s  + 3 s  + 7 s + 5)


c(t)=
                      /             sin(2 t) \
            3 exp(-t) | cos(2 t) - -------- |
                      \                 3    /
  3/5 - ----------------------------------
                            5
```

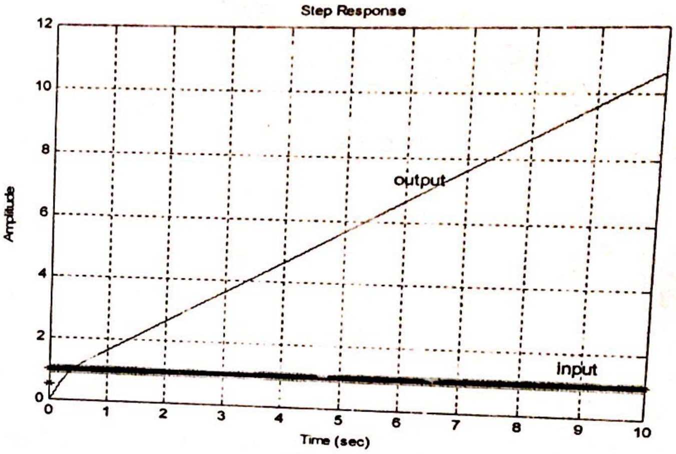
Example: MATLAB Step Response

Use MATLAB to find the step response of the following transfer function:

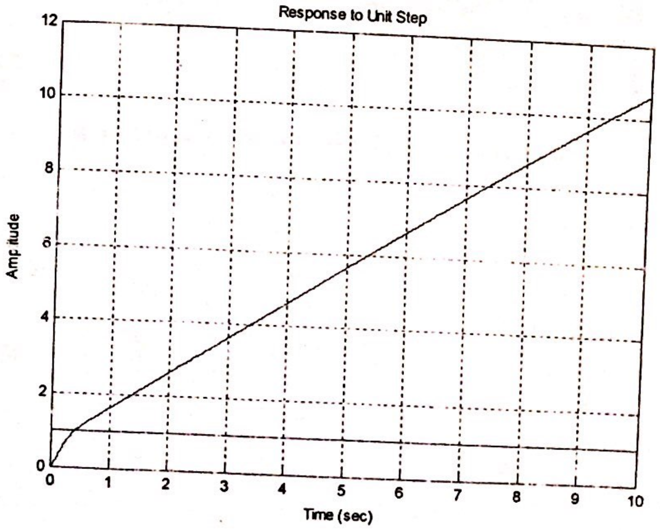
$$F(s) = \frac{79s^2 + 916s + 1000}{s(s+10)^3}$$

We can separate out our numerator and denominator polynomials as such:

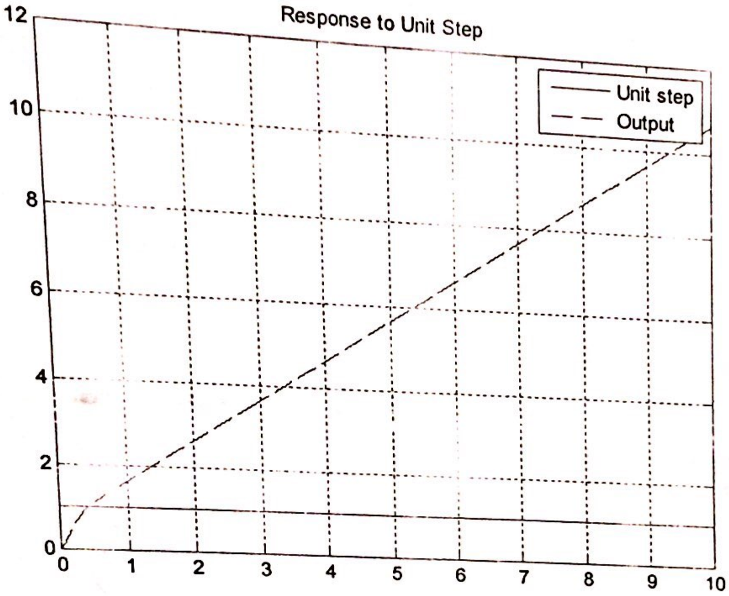
```
num = [79 916 1000];
den = [1 30 300 1000 0];      %den= poly([0 -10 -10 -10])
sys = tf(num, den)
T = 0:0.001:10
step(sys,  T)
hold on
plot(T,heaviside(T),'b')
gtext('input')
gtext('output')
```



```
num = [79 916 1000];
den = [1 30 300 1000 0];      %den= poly([0 -10 -10 -10])
T = 0:0.01:10;                % simulation time = 10 seconds
U = ones(size(T));            %  input unit step
sys = tf(num,den);            % construct a system model
lsim(sys, U, T)               % simulate and plot the response (the
output)
title('Response to unit step')
```



```
T = 0:0.01:10;                          % simulation time = 10 seconds
U = ones(size(T));                      %  input unit step
num = [79 916 1000];
den = [1 30 300 1000 0];                %den= poly([0 -10 -10 -10])
sys= tf(num,den)
y= lsim(sys, U, T)        % simulate and plot the response (the output)
plot(T,U,'-r',T,y,'--b')
title('Response to Unit Step')
legend('Unit step','Output')
grid
```



Ex

For each of the following transfer function, write the corresponding differential equation.

$$\frac{X(s)}{F(s)} = \frac{s+2}{s^3+11s^2+12s+18}$$

Cross multiplying, $(s^3+8s^2+9s+15)X(s) = (s+2)F(s)$

Taking the inverse Laplace transform, $\frac{d^3x}{dt^3} + 8\frac{d^2x}{dt^2} + 9\frac{dx}{di} + 15x = \frac{df(t)}{dt} + 2f(t)$